

## **The Mechanics of Axisymmetric Indentation Revisited**

**by George A. Gazonas and Raymond A. Wildman**

**ARL-TR-6528**

**July 2013**

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**George A. Gazonas and Raymond A. Wildman**

**Weapons and Materials Research Directorate, ARL**

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## 1. Introduction

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The problem of axisymmetric indentation of an elastic half-space by a rigid flat-ended cylindrical punch (the Boussinesq problem) is summarized in this report. Exact analytical solutions to elastic indentation problems are important for verification of numerical codes capable of modeling such phenomena (2, 3), and for modeling the onset of permanent deformation and fracture (5), a topic of active research in the Materials and Manufacturing Sciences Division of the U.S. Army Research Laboratory.

Boussinesq (6) and Love (7) were among the first to consider such a problem, but Sneddon (4, 8) solved the mixed boundary value problem for the rigid flat-ended punch in cylindrical polar coordinates by using the Hankel transform to convert the governing equation, i.e., the biharmonic equation  $\nabla^4\psi = 0$ , into a set of dual integral equations solvable by methods introduced by Titchmarsh (9). The generality of Titchmarsh's methods (9) were such that they were also used by Busbridge (10), and Tranter (11) to derive solutions to the electrified disk problem important in potential field theory,  $\nabla^2\phi = 0$ . Other powerful methods of analysis rely on using the Papkovitch-Neuber solution (Galerkin vector), which for body-force free axisymmetric problems involves determining two scalar harmonic functions solvable by Hankel transforms (12). Hankel transforms and dual integral equations have been used extensively by a number of researchers to solve axisymmetric boundary value problems in fluid (13), poroelastic (14), piezoelectric (15), and magneto-electroelastic (16) media.

We focus our study on the fully three-dimensional, axisymmetric indentation problem despite the long history of developments in the two-dimensional theory of mixed boundary value problems, known as Riemann-Hilbert problems, and their solution via the Plemelj (17) formulae;<sup>1</sup> two-dimensional half-plane solutions for rigid flat punch indentation problems are decidedly non-physical,<sup>2</sup> since vertical surface displacements become unbounded away from the punch. This result has been obtained by both England (21; page 70, equation (3.67)) using complex variable methods, and Sneddon (1; page 433, equation (115)) using Fourier transform methods.

During the course of our literature survey on axisymmetric indentation, we encountered many typographical errors or misprints of equations that have been unwittingly propagated through the

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<sup>1</sup>Parton and Perlin (18) refer to these equations as the Sokhotskii-Plemelj formulae, and Gakhov (19) refers to these equations as the Sokhotskii formulae, since Sokhotskii derived them first in his 1873 dissertation.

<sup>2</sup>Solutions to boundary value problems that are non-physical may be more useful for code verification than exact physical solutions with field singularities; see, e.g., the method of manufactured solutions in Oberkampf and Roy (20).

literature by various authors; we have identified these errors in this report for readers interested in such analysis. Furthermore, the original works of Sneddon (*I*, 4, 8) lacked analytical expressions for the vertical displacements within the half-space region; indeed, only expressions for the vertical displacements of the surface of the half-space are given,<sup>3</sup> and we wondered why the more general half-space solution was omitted by Sneddon. The vertical displacement solution in the entire half-space is important for verification of numerical codes. We traced the probable cause of the omission to the unevaluated  $J_0^0$  term appearing in the expression for the vertical displacement  $u_z$ .<sup>4</sup> In general, the  $J_n^m$  terms can be defined (*I*) by taking the imaginary  $\Im$  part of a Laplace transform involving  $m^{\text{th}}$ -order Bessel functions  $J_m(p\rho)$

$$J_n^m = \Im \int_0^\infty p^{n-1} e^{-p(\zeta-i)} J_m(p\rho) dp \quad . \quad (1)$$

Explicit solutions to equation 1 can be obtained by taking the imaginary part of a more general result found in Sneddon's textbook (*I*; page 514, appendix A),

$$J_n^m = \Im \left( 2^{-m} \rho^m (\zeta - i)^{-m-n} \Gamma(m+n) {}_2\tilde{F}_1 \left( \frac{m+n}{2}, \frac{1}{2}(m+n+1); m+1; -\frac{\rho^2}{(\zeta-i)^2} \right) \right), \quad (2)$$

where  $(\rho, \zeta) > 0$ ,  $\Re(m+n) > 0$ , and  ${}_2\tilde{F}_1$  is the regularized hypergeometric function  ${}_2\tilde{F}_1 = {}_2F_1(a, b; c; z)/\Gamma(c)$ . According to Watson (22; page 384), both Hankel (23) and Gegenbauer (24) “proved that  $\Re(m+n) > 0$ , to secure convergence at the origin.” More direct is the observation that for the  $J_0^0$  term, if  $m = n = 0$ , then  $\Gamma(m+n) = (m+n-1)!$  is undefined in equation 2. We also found in the Tables of Integral Transforms from the Bateman Manuscript Project (25; page 182, equation (5)) that a solution to the singular integral (equation 1) does not exist for  $m = n = 0$ ,

$$J_0^m = \Im \int_0^\infty \frac{e^{-p(\zeta-i)} J_m(p\rho)}{p} dp = \Im \left( \frac{(\zeta-i)^{-m} \rho^m \left( 1 + \sqrt{1 + \frac{\rho^2}{(\zeta-i)^2}} \right)^{-m}}{m} \right) \quad m > 0 \quad . \quad (3)$$

However, from the same work (25), one can find an explicit representation for the  $J_0^0$  term given

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<sup>3</sup>See e.g., Sneddon's textbook (*I*; page 461, the equation is not numbered).

<sup>4</sup>See e.g., Sneddon's textbook (*I*; page 459, equation (44)), and reproduced for convenience in equation 5 of this report.



on page 101, equation (17) in terms of a Fourier sine transform,

$$J_0^0 = \int_0^\infty \frac{e^{-p\zeta} \sin(p) J_0(p\rho)}{p} dp = \sin^{-1} \left( \frac{2}{\sqrt{\zeta^2 + (\rho - 1)^2} + \sqrt{\zeta^2 + (\rho + 1)^2}} \right) . \quad (4)$$

Thus, a formal relationship should be derivable between the solution for  $J_0^0$  given in equation 4 and equation 3 evaluated at  $m = 0$ ; we derive such a relationship in this report through application of L'Hôpital's rule to equation 3. In fact, we derive new expressions for all the  $J_n^m$  terms that appear as stress and displacement components in Sneddon's solution to the axisymmetric indentation problem (*I*). The new analytical solutions are used to verify our peridynamics numerical code (2, 3), and illustrate principal stress trajectories in an elastic half-space under the action of a rigid cylindrical punch. Sneddon's classic textbook (*I*) was published only three years before the publication of the Bateman treatise (25) so the reason why Sneddon was not aware of, or did not publish the result given by equation 4, remains an open question and was in fact another motivation for this research.

## 2. The Mechanics of Indentation

The solutions for the displacements and stresses in an elastic half-space under the action of a rigid flat-ended cylindrical punch that appear in Sneddon's textbook (*I*) are listed below:

$$\begin{aligned} u_r &= -\frac{2\mu\epsilon \left( J_0^1 - \frac{(\lambda+\mu)}{\mu} \zeta J_1^1 \right)}{\pi(\lambda + 2\mu)} , \\ u_z &= \frac{2\epsilon \left( \frac{(\lambda+\mu)}{\lambda+2\mu} \zeta J_1^0 + J_0^0 \right)}{\pi} , \\ \sigma_z &= -\frac{4\mu\epsilon (\zeta J_2^0 + J_1^0) (\lambda + \mu)}{\pi a(\lambda + 2\mu)} , \\ \tau_{rz} &= -\frac{4\zeta J_2^1 \mu\epsilon (\lambda + \mu)}{\pi a(\lambda + 2\mu)} , \\ \sigma_\theta &= -\frac{4J_1^0 \lambda \mu\epsilon}{\pi a(\lambda + 2\mu)} - \frac{4\mu^2\epsilon \left( J_0^1 - \frac{(\lambda+\mu)}{\mu} \zeta J_1^1 \right)}{\pi a\rho(\lambda + 2\mu)} , \\ \sigma_r + \sigma_\theta &= -\frac{4\mu\epsilon (J_1^0(2\lambda + \mu) - \zeta J_2^0(\lambda + \mu))}{\pi a(\lambda + 2\mu)} . \end{aligned} \quad (5)$$

In these equations, the dimensionless radial coordinate  $\rho = r/a$  is the physical radial distance  $r$  normalized by the indenter (punch) radius  $a$ , and the dimensionless depth coordinate  $\zeta = z/a$  is the physical depth  $z$  normalized by the punch radius  $a$ .  $\epsilon$  is the indentation depth, and the Lamé parameters are given by  $\lambda$  and  $\mu$ . The radial stress  $\sigma_r$  is not given in Sneddon's textbook (1) but can be derived readily by subtraction of  $\sigma_\theta$  from  $\sigma_r + \sigma_\theta$  from equations 5 resulting in,

$$\sigma_r = \frac{4\mu\epsilon(J_0^1\mu - (\lambda + \mu)(\rho(J_1^0 - \zeta J_2^0) + \zeta J_1^1))}{\pi a \rho(\lambda + 2\mu)} \quad , \quad (6)$$

where the  $J_n^m$  terms are defined as singular integrals involving Bessel functions  $J_m(p\rho)$ ,

$$J_n^m = \int_0^\infty p^{n-1} e^{-\zeta p} \sin(p) J_m(p\rho) dp \quad , \quad (7)$$

or as Laplace transforms involving Bessel functions, i.e., the imaginary part of the singular integral,

$$J_n^m = \Im \int_0^\infty p^{n-1} e^{-p(\zeta-i)} J_m(p\rho) dp \quad . \quad (8)$$

Explicit analytical expressions for a number of specializations of the integral in equation 8 can be found in the Tables of Integral Transforms from the Bateman Manuscript Project (25; page 182; equation (5)) for example, if  $m > 0$  and  $n = 0$ ,

$$J_0^m = \Im \int_0^\infty p^{-1} e^{-p(\zeta-i)} J_m(p\rho) dp = \Im \left( \frac{(\zeta - i)^{-m} \rho^m \left(1 + \sqrt{1 + \frac{\rho^2}{(\zeta-i)^2}}\right)^{-m}}{m} \right) . \quad (9)$$

Thus, for  $m = 1$ , equation 9 becomes

$$J_0^1 = \Im \left( \frac{\rho}{(\zeta - i)(1 + H(\rho, \zeta))} \right) \quad , \quad (10)$$

where,  $H(\rho, \zeta) = \sqrt{1 + \frac{\rho^2}{(\zeta-i)^2}}$ . However, for  $m = 0$ , care must be taken for evaluation of the integral in equation 9 via a limit process, i.e.,

$$J_0^0 = \lim_{m \rightarrow 0} J_0^m = \lim_{m \rightarrow 0} \frac{f(m)}{g(m)} = \lim_{m \rightarrow 0} \Im \left( \frac{(\zeta - i)^{-m} \rho^m (1 + H(\rho, \zeta))^{-m}}{m} \right) . \quad (11)$$

In evaluation of the limit in equation 11, since  $f(m) = \Im(1) = g(m) = 0$  as  $\Re(m) \rightarrow 0$ , and since the limit of the ratio  $f'(m)/g'(m)$  as  $\Re(m) \rightarrow 0$  exists, we may employ L'Hôpital's rule for the evaluation of  $J_0^0$ ,

$$J_0^0 = \lim_{m \rightarrow 0} \frac{f(m)}{g(m)} = \lim_{m \rightarrow 0} \frac{f'(m)}{g'(m)} . \quad (12)$$

The derivative of the numerator and denominator of equation 11 results in

$$J_0^0 = \lim_{m \rightarrow 0} \frac{-\Im \left( (\zeta - i)^{-m} \rho^m (\log(1 + H(\rho, \zeta)) + \log(\zeta - i) - \log(\rho)) (1 + H(\rho, \zeta))^{-m} \right)}{1} . \quad (13)$$

Thus,

$$J_0^0 = -\Im (\log(1 + H(\rho, \zeta)) + \log(\zeta - i) - \log(\rho)) , \quad (14)$$

or

$$J_0^0 = -\Im \left( \log \frac{(\zeta - i)(1 + H(\rho, \zeta))}{\rho} \right) . \quad (15)$$

Alternatively, since  $z = |z| e^{i \arg(z)}$ , then  $\log(z) = \ln |z| + i \arg(z)$ , and  $\Im(\log(z)) = \arg(z)$ <sup>5</sup> resulting in

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<sup>5</sup>Multivalued  $\arg(z)$  is defined by  $\arg(z) = \{\text{Arg}(z) + 2\pi k : k \in \mathbb{Z}\}$ ,  $-\pi < \text{Arg}(z) \leq \pi$  .

$$J_0^0 = -\arg\left(\frac{(\zeta - i)(1 + H(\rho, \zeta))}{\rho}\right) . \quad (16)$$

A more direct means for deriving equation 16 (26), can be seen by recognizing that since

$$z = |z| e^{i \arg(z)} = |z| e^{i\theta} , \quad (17)$$

then equation 11 can be written succinctly as

$$\lim_{m \rightarrow 0} \Im\left(\frac{z^m}{m}\right) = \lim_{m \rightarrow 0} \theta |z|^m \frac{\sin(m\theta)}{m\theta} = \lim_{m \rightarrow 0} \theta |z|^m \text{sinc}(m\theta) = \theta = \arg(z) . \quad (18)$$

Interestingly, we later became aware of an expression for  $J_0^0$  in Barquins and Maugis (27; page 355; appendix 2), who obtained it from an earlier work of Dahan<sup>6</sup> (28), although the steps taken for its derivation do not appear in either work. The expression that appears in Barquins and Maugis (27) using Sneddon's ( $I$ ) notation is

$$J_0^0 = \frac{\pi}{2} - \tan^{-1}\left(\frac{\sqrt{\sqrt{(\zeta^2 + \rho^2 - 1)^2 + 4\zeta^2} + \zeta^2 + \rho^2 - 1}}{\sqrt{2}}\right) . \quad (19)$$

Dahan (28) refers to the earlier work of Gerrard and Harrison<sup>7</sup> (30), where in fact, an expression for  $J_0^0$  can be extracted by specializing their general results for an anisotropic elastic half-space to an isotropic elastic half-space under vertical displacement by a rigid, flat-ended cylindrical punch (see e.g., pages 12, 14, and 15 of Gerrard and Harrison (30)). The expression for  $J_0^0$  that we have extracted from Gerrard and Harrison (30) using Sneddon's ( $I$ ) notation is

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<sup>6</sup>Dahan (28) refers to his thesis (29) as a possible source for derivation, which is inaccessible to us.

<sup>7</sup>Gerrard and Harrison (30) refer to an earlier reference from Koning (31) where we find all of Sneddon's ( $I$ ) expressions (27) listed without reference; in addition, a correct expression for  $J_0^0 = \sin^{-1}\left(\sqrt{\zeta^2 + \rho^2 + 1 - \sqrt{(\zeta^2 + \rho^2 + 1)^2 - 4\rho^2/(\sqrt{2}\rho)}}\right)$  appears without derivation.

$$J_0^0 = \sin^{-1} \left( \frac{2}{\sqrt{\zeta^2 + (\rho - 1)^2} + \sqrt{\zeta^2 + (\rho + 1)^2}} \right) . \quad (20)$$

Equation 20 is identical to that found in the Bateman Manuscript Project (25), i.e., equation 4 herein, and subsequently found in Fabrikant (32; page 221, equation (54)). Fabrikant (33) also applied the theory of the potential to solve a number of mixed boundary value problems in mechanics, including the bonded/unbonded cylindrical punch problem in both isotropic and transversely isotropic elastic media (32). Using the identity  $\sin^{-1} \left( \frac{1}{\sqrt{x^2+1}} \right) = \frac{\pi}{2} - \tan^{-1}(x), x > 0$  one can verify that equation 19 is identical to equation 20. Our own result for  $J_0^0$ , given by equation 15 or equation 16, can be shown to be equivalent to that of Gerrard and Harrison (30) in equation 20 using an identity from Churchill's text (34; page 62, section 29), that relates the inverse sine function to a logarithmic function,

$$\sin^{-1}(z) = -i \log \left( \sqrt{1 - z^2} + iz \right) = -i \log \left( \frac{(\zeta - i) \left( 1 + \sqrt{1 + \frac{\rho^2}{(\zeta - i)^2}} \right)}{\rho} \right) , \quad (21)$$

and if we solve equation 21 for  $z$ , we find that  $z = -\frac{1+i\zeta}{\rho}$ . Thus, our result in equation 16 follows readily by simplification of  $\sin^{-1} \left( -\frac{1+i\zeta}{\rho} \right)$ . A plot of the  $J_0^0$  term is illustrated in figure 1.

Barquins and Maugis (27) have identified several typographical errors in the literature by noting "...that  $J_2^0$  and  $\sigma_\theta$  are misprinted in..." Sneddon's paper (4), i.e., equations (3) and (16) in that work<sup>8</sup>; we also discovered another error in the term for  $u_r$  appearing in equation (8) of Sneddon's paper<sup>9</sup> (4); this error was also incorrectly transcribed to appendix 2, equation (34) of Barquins and Maugis (27). These typographical errors are corrected in Sneddon's textbook (1).

Proceeding in a similar fashion by using the results from (25; page 182, equation (7)) for  $m = 1, n = 2$ , equation 8 becomes

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<sup>8</sup>  $J_0^1$  should be  $J_1^0$  in the first term, and  $J_2^1$  should be  $J_1^2$  in the last term of equation (3) in Sneddon's paper (4); the transcendental argument  $\sin \left( \frac{3(\varphi - \theta)}{2} \right)$  should appear as  $\sin \left( \frac{3\varphi}{2} - \theta \right)$  in equation (16) of Sneddon's paper (4).

<sup>9</sup> The  $(J_0^1 - \frac{(\lambda + \mu)}{\lambda} \zeta J_1^1)$  term should appear as  $(J_0^1 - \frac{(\lambda + \mu)}{\mu} \zeta J_1^1)$  in equation (8) of Sneddon's paper (4).

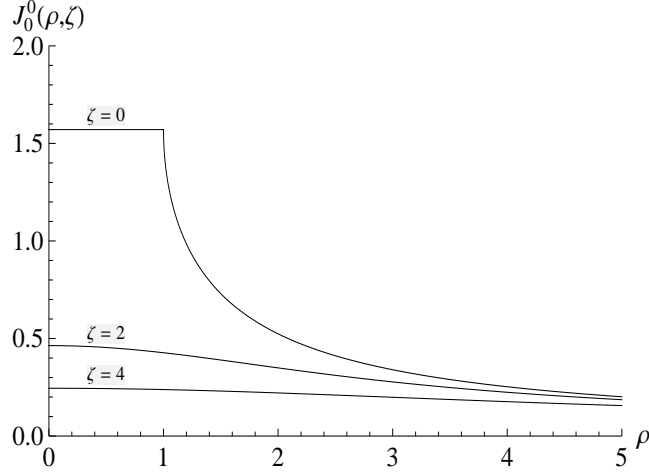


Figure 1. Plot of the  $J_0^0$  term given by equation 16, 19, or 20.

$$J_2^1 = \Im \int_0^\infty p e^{-p(\zeta-i)} J_1(p\rho) dp = \Im \left( \frac{2\rho\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}(\rho^2 + (\zeta-i)^2)^{3/2}} \right) = \Im \left( \frac{\rho}{(\zeta-i)^3 H(\rho, \zeta)^3} \right) . \quad (22)$$

Thus, using the result from the Bateman Manuscript Project (25; page 182, equation (1)) for  $m = 1, n = 1$ , equation 8 becomes

$$J_1^1 = \Im \int_0^\infty e^{-p(\zeta-i)} J_1(p\rho) dp = \Im \left( \frac{\rho}{(\zeta-i)^2 H(\rho, \zeta)(H(\rho, \zeta) + 1)} \right) \quad (23)$$

and using the result from the Bateman Manuscript Project (25; page 182, equation (1)) for  $m = 0, n = 1$ , equation 8 becomes

$$J_1^0 = \Im \int_0^\infty e^{-p(\zeta-i)} J_0(p\rho) dp = \Im \left( \frac{1}{(\zeta-i)H(\rho, \zeta)} \right) . \quad (24)$$

Finally, using the result from the Bateman Manuscript Project (25; page 182, equation (2)) for  $m = 0, n = 2$ , equation 8 becomes

$$J_2^0 = \Im \int_0^\infty p e^{-p(\zeta-i)} J_0(p\rho) dp = \Im \left( \frac{1}{(\zeta-i)^2 H(\rho, \zeta)^3} \right) . \quad (25)$$

Our results for the  $J_n^m$  terms necessary for determining the stress and displacement fields (equations 5 and 6) in an elastic half-space under the action of a rigid flat-ended cylindrical punch are written succinctly as

$$\begin{aligned} J_1^0 &= \Im \left( \frac{1}{(\zeta-i) H(\rho, \zeta)} \right) , \\ J_2^0 &= \Im \left( \frac{1}{(\zeta-i)^2 H(\rho, \zeta)^3} \right) , \\ J_1^1 &= \Im \left( \frac{\rho}{(\zeta-i)^2 H(\rho, \zeta) (H(\rho, \zeta) + 1)} \right) , \\ J_2^1 &= \Im \left( \frac{\rho}{(\zeta-i)^3 H(\rho, \zeta)^3} \right) , \\ J_0^1 &= \Im \left( \frac{\rho}{(\zeta-i) (1 + H(\rho, \zeta))} \right) , \\ J_0^0 &= -\arg \left( \frac{(\zeta-i) (1 + H(\rho, \zeta))}{\rho} \right) . \end{aligned} \quad (26)$$

The results summarized in equation 26 are derived using the Tables of Integral Transforms from the Bateman Manuscript Project (25). If we use instead, Table VII. Hankel Transforms on page 528 of Sneddon's textbook (1), we can verify the results summarized in equation 26. Using the Hankel transform tables found in Sneddon's textbook (1), however, we find that the two terms  $J_0^1$  and  $J_1^1$  differ in form, but are equivalent to those appearing in equation 26. These are given by  $J_0^1 = \Im \left( \frac{(\zeta-i)(H(\rho, \zeta)-1)}{\rho} \right) = \frac{1}{\rho} \Im((\zeta-i)(H(\rho, \zeta)-1))$  and  $J_1^1 = \Im \left( \frac{H(\rho, \zeta)-1}{\rho H(\rho, \zeta)} \right) = -\frac{1}{\rho} \Im \left( \frac{1}{H(\rho, \zeta)} \right)$ .

Our  $J_n^m$  expressions in terms of dimensionless coordinates  $\rho$  and  $\zeta$  differ in form, but are equivalent<sup>10</sup> to those given in Sneddon's textbook<sup>11</sup> (1) and Barquins and Maugis (27), viz.,

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<sup>10</sup>We have verified this numerically, to machine precision, by comparing all  $J_n^m$  terms in equation 26 and equation 27.

<sup>11</sup>All equations 27 appear originally in Sneddon's textbook (1), except for the  $J_0^0$  term, which appears explicitly in Barquins and Maugis (27). The  $J_2^1$  term is misprinted in Sneddon's textbook (1; page 462, equation (56)), as it omits the  $\rho$  that multiplies the  $\sin \left( \frac{3\phi}{2} \right)$  term.

$$\begin{aligned}
J_1^0 &= \frac{\sin\left(\frac{\varphi}{2}\right)}{\sqrt{R}} \quad , \\
J_2^0 &= \frac{r \sin\left(\frac{3\varphi}{2} - \theta\right)}{R^{3/2}} \quad , \\
J_1^1 &= \frac{r \sin\left(\theta - \frac{\varphi}{2}\right)}{\rho\sqrt{R}} \quad , \\
J_2^1 &= \frac{\rho \sin\left(\frac{3\varphi}{2}\right)}{R^{3/2}} \quad , \\
J_0^1 &= \frac{1 - \sqrt{R} \sin\left(\frac{\varphi}{2}\right)}{\rho} \quad , \\
J_0^0 &= \frac{\pi}{2} - \tan^{-1} \left( \frac{\sqrt{\zeta^2 + \rho^2 + R - 1}}{\sqrt{2}} \right) \quad ,
\end{aligned} \tag{27}$$

where  $R = \sqrt{(\rho^2 + \zeta^2 - 1)^2 + 4\zeta^2}$  as before,  $r^2 = 1 + \zeta^2$ ,  $\zeta \tan(\theta) = 1$ , and  $(\rho^2 + \zeta^2 - 1) \tan(\phi) = 2\zeta$ .

Sneddon's solutions (equation 5) can be rewritten more succinctly using the substitution,  $\lambda = \frac{2\mu\nu}{1-2\nu}$ , and by normalizing the stress components by the mean pressure  $p_m = \frac{8\mu\epsilon(\lambda+\mu)}{\pi a(\lambda+2\mu)}$ , and the displacement components by the indentation depth  $\epsilon$ , resulting in

$$\begin{aligned}
\frac{u_r(\rho, \zeta)}{\epsilon} &= \frac{(\zeta J_1^1 - (1 - 2\nu)J_0^1)}{\pi(1 - \nu)} \quad , \\
\frac{u_z(\rho, \zeta)}{\epsilon} &= \frac{(2(1 - \nu)J_0^0 + \zeta J_1^0)}{\pi(1 - \nu)} \quad , \\
\frac{\tau_{rz}(\rho, \zeta)}{p_m} &= -\frac{\zeta J_2^1}{2} \quad , \\
\frac{\sigma_z(\rho, \zeta)}{p_m} &= -\frac{1}{2} (\zeta J_2^0 + J_1^0) \quad , \\
\frac{\sigma_\theta(\rho, \zeta)}{p_m} &= -\frac{1}{2\rho} [2\nu\rho J_1^0 - \zeta J_1^1 + (1 - 2\nu)J_0^1] \quad , \\
\frac{\sigma_r(\rho, \zeta)}{p_m} &= -\frac{1}{2\rho} [J_1^0\rho - \zeta(\rho J_2^0 - J_1^1) - (1 - 2\nu)J_0^1] \quad .
\end{aligned} \tag{28}$$

Thus, the dimensionless analytical solutions for displacements and stresses in an elastic half-space (equation 28) are functions of Poisson's ratio  $\nu$ , and substantiates the choice of  $\nu$  alone



to describe the elastic properties of the half-space in the dimensional analysis for Hertzian cone cracks developed in Gazonas et al. (5); these equations are corroborated by those found in Barquins and Maugis (27), with the exception of the expression for  $u_r$ , misprinted in both Barquins and Maugis (27) and Sneddon (4).

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### 3. Peridynamics Code Verification

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In the previous section, we have shown how to derive a new explicit expression for  $J_0^0$  necessary for determining the vertical displacements in an elastic half-space (equation 28); the solution for the vertical displacements,

$$\frac{u_z(\rho, \zeta)}{\epsilon} = -\frac{2 \arg \left( (\zeta - i) \left( 1 + \sqrt{1 + H(\rho, \zeta)} \right) \right) + \frac{\zeta}{\nu-1} \Im \left( \frac{1}{(\zeta-i)\sqrt{1+H(\rho, \zeta)}} \right)}{\pi}, \quad (29)$$

of the half-space is illustrated in figure 2 (solid lines), where we verify a numerical peridynamic solution (dashed lines) at dimensionless depths,  $\zeta = 0, \zeta = 1.7$ , and  $\zeta = 5.4$ . A 100 x 100 node grid spacing was used for all axisymmetric peridynamic simulations. The perfectly matched layer (PML) developed by Wildman and Gazonas (3) for 2-D peridynamics, is employed in this axisymmetric solution at finite radial and vertical distances  $\rho = \zeta \approx 8$  within the computational domain to simulate the zero vertical displacement condition that exists at infinity. The solution for the radial displacements,

$$\frac{u_r(\rho, \zeta)}{\epsilon} = \frac{(1 - 2\nu) \Im \left( (\zeta - i) \left( -1 + \sqrt{1 + H(\rho, \zeta)} \right) \right) + \zeta \Im \left( \frac{1}{\sqrt{1+H(\rho, \zeta)}} \right)}{\pi(\nu - 1)\rho}, \quad (30)$$

of the half-space is illustrated in figure 3 (solid lines), where we verify our numerical peridynamic solution (dashed lines) at dimensionless depths,  $\zeta = 0, \zeta = 1.7$ , and  $\zeta = 5.4$ . The analytical radial surface displacements of the half-space are generally non-monotonic and the computed peridynamic results accurately reproduce the theoretical trends as a function of depth within the half-space; we note, however, that the computed radial displacements undergo a small spatial wrinkle at  $\rho = 1$  at the corner of the rigid punch, and we surmise that this is related to singularities present in the stress and strain fields that may influence the non-singular

displacement fields. Since we do not observe such behavior at  $\rho = 1$  in the computed vertical displacement field illustrated in figure 2, it is also possible that this is a result of a solution that has not fully converged.

Using the theory of the potential, Fabrikant (32; page 221, equations (53) and (54)) has independently derived analytical solutions for both the bonded and unbonded punch problem; Fabrikant's solutions are to within machine precision, numerically equivalent to the new solutions  $u_z$  in equation 29 and  $u_r$  in equation 30. The mean pressure  $p_m = \frac{2\epsilon}{\pi a} \frac{E}{1-\nu^2}$  induced in the elastic half-space with rigid punch radius  $a = 5$  mm, indentation depth  $\epsilon = 0.5$  mm, Young's modulus,  $E = 65$  GPa, and Poisson's ratio,  $\nu = 0.20$  is 4.3 GPa.

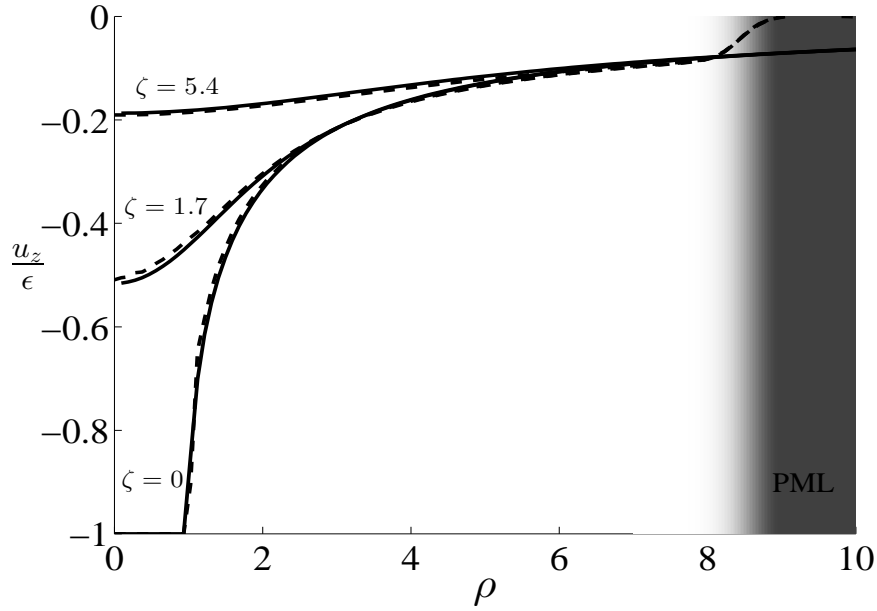


Figure 2. Verification of our peridynamics code results (dashed lines) using the analytical solution (solid lines; equation 29) for normalized vertical displacements  $\frac{u_z}{\epsilon}$  in an isotropic elastic half-space,  $\nu = 0.2$ , due to a rigid flat-ended cylindrical punch; the mean pressure induced in the half-space is  $p_m = 4.3$  GPa.

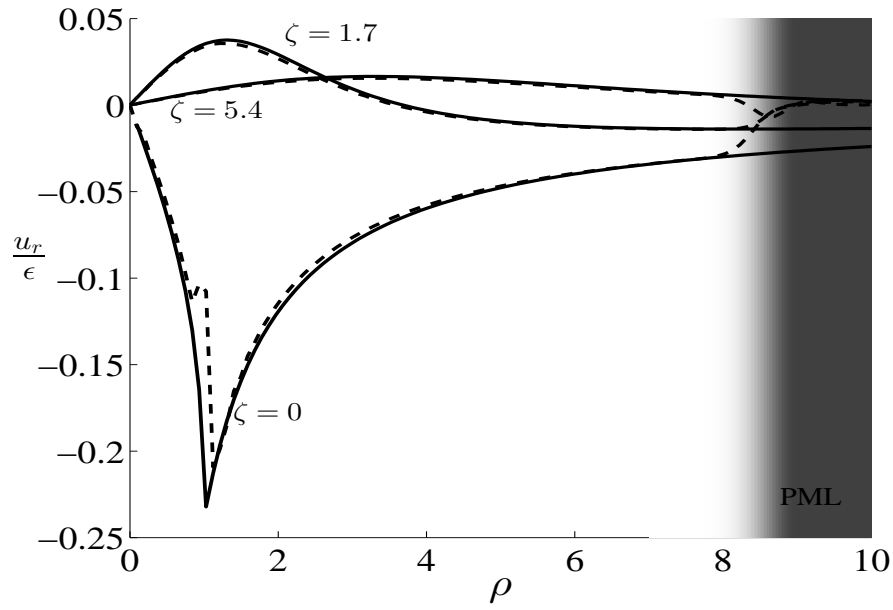


Figure 3. Verification of our peridynamics code results (dashed lines) using the analytical solution (solid lines; equation 30) for normalized radial displacements  $\frac{u_r}{\epsilon}$  in an isotropic elastic half-space,  $\nu = 0.2$ , due to a rigid flat-ended cylindrical punch.

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## 4. Stress Trajectories

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Stress trajectories or isostatics (principal stress directions in the plane  $\theta = \text{constant}$ ) for the problem of the half-space indented by a rigid flat-ended cylindrical punch first appear in Sneddon (4) and can be determined by solving

$$\tan(2\alpha) = \frac{2\tau_{rz}}{\sigma_r - \sigma_z} \quad (31)$$

for  $\alpha$  inclined at angles  $\alpha$  and  $\alpha + \pi/2$  to the radial axis,  $r$ . Use of the auxiliary relationship in dimensionless coordinates,  $\tan(\alpha) = d\zeta/d\rho$ , we arrive at a differential equation for solution of the isostatics for maximum and minimum principal stress directions in a half-space under the action of a rigid flat-ended cylindrical punch:

$$\frac{d\zeta}{d\rho} = -\frac{\sigma_r - \sigma_z}{2\tau_{rz}} \pm \sqrt{1 + \left(\frac{\sigma_r - \sigma_z}{2\tau_{rz}}\right)^2} \quad (32)$$

Substitution of stress components  $\sigma_r$ ,  $\sigma_z$ , and  $\tau_{rz}$  from equation 28 into equation 32 results in

$$\frac{d\zeta}{d\rho} = \frac{2\zeta\rho J_2^0 + (1 - 2\nu)J_0^1 - \zeta J_1^1}{2\zeta\rho J_2^1} + \sqrt{1 + \frac{(2\zeta\rho J_2^0 + (1 - 2\nu)J_0^1 - \zeta J_1^1)^2}{4\zeta^2\rho^2(J_2^1)^2}} \quad (33)$$

To determine the isostatics for the rigid punch indentation problem, equation 33 (for the  $\sigma_3$  trajectory) and its negative reciprocal (for the  $\sigma_1$  trajectory) must be numerically integrated with a computational software program like Mathematica 8.0 (35). The maximum shearing stress trajectories bisect the angle made between the orthogonal principal stress trajectories, thus, one set of maximum shearing stress trajectories can be determined from the equation

$$\frac{d\zeta}{d\rho} = \frac{q - 1}{q + 1} \quad (34)$$

and an orthogonal set of maximum shearing stress trajectories can be determined from

$$\frac{d\zeta}{d\rho} = \frac{q+1}{1-q} \quad (35)$$

where  $q$  is given by the right-hand side of equation 33. The isostatics and maximum shear stress trajectories have been determined numerically using Mathematica 8.0 (35) and these curves are illustrated in figure 4. In general, our numerically computed trajectories compare quite well with Sneddon's (4) hand-drawn curves, generated "by tracing two sets of curves which are at every point tangential to the direction of the two principal stresses in the plane." Our  $\sigma_1$  trajectories (solid black lines in figure 4), however, tend to parallel the free surface in regions outside the contact area, a result shown also in Barquins and Maugis (27; page 342, figure 4), whereas Sneddon's curves (4; page 38, figure 6) are in error, and orthogonal to the free surface in such regions. Because our maximum shear stress trajectories (dashed lines) should intersect the free surface, i.e. along the  $\rho$ -axis at  $45^\circ$ , as they do along the  $\zeta$ -axis, they are in error here, as we were unable to numerically integrate equation 35 along the  $\rho$ -domain in regions where the trajectories become vertical and have undefined slope,  $\frac{d\zeta}{d\rho} = \infty$ ; we attempted to re-couch equation 35 in terms of polar coordinates to eliminate the numerical singularity, but were still unable to obtain a solution in this region.

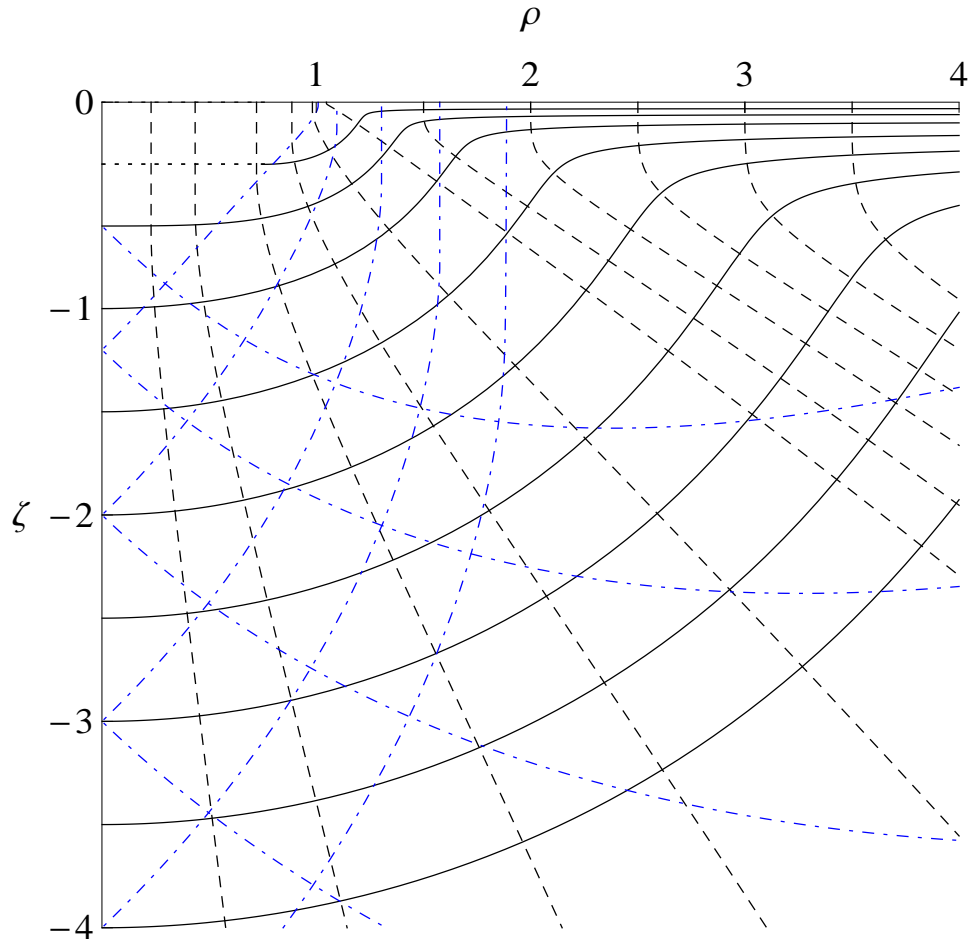


Figure 4. Axisymmetric isostatics in an elastic medium,  $\nu = 0.1679$ , indented by a flat-ended cylindrical punch;  $\sigma_1$  stress trajectories (solid black lines),  $\sigma_3$  stress trajectories (dashed black lines), and principal shear stress trajectories (orthogonal dot-dashed blue lines).

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## 5. Conclusions

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In this report, we have derived new expressions (equation 26) for the  $J_n^m$  terms that appear as stress and displacement components in Sneddon's solution to the axisymmetric indentation problem (1). The new solutions are used to verify a newly developed peridynamics numerical code (2, 3). The analytical solutions for the normal and radial displacements of an elastic half-space under the action of a rigid cylindrical punch compare well with the elastic deformations from peridynamic simulations. Principal stress and shear trajectories in the elastic half-space under the action of a rigid cylindrical punch were derived by numerical integration of equation 33; our results generally compare well with Sneddon's hand-drawn curves, but Sneddon's  $\sigma_1$  trajectories (4) are in error near the surface of the half-space and outside the region of indentation.

Sneddon's classic textbook (1) was published only three years before the publication of the Bateman Manuscript Project treatise (25), so the reason why Sneddon was not aware of, or did not publish, any result for the  $J_0^0$  term, given by equation 4 (page 101, equation (17) of that treatise), remains an open question; the omission of the  $J_0^0$  term, and hence expressions for the vertical displacements within the half-space, is also apparent in more recent contact mechanics textbooks (12, 36). Finally, we derived a new expression for the  $J_0^0$  term with a relatively straightforward application of L'Hôpital's rule and have shown that it is analytically equivalent to the results of Barquins and Maugis (27) (equation 19), Gerrard and Harrison (equation 20), and that found in Bateman's treatise (equation 4). The new analytical results for the remaining  $J_n^m$  terms (equation 26) compare exactly to within machine precision<sup>12</sup> with those from Sneddon's textbook (1) (equation 27) for a number of half-space depths  $\zeta$  and radial distances  $\rho$ .

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<sup>12</sup>This result is shown analytically for the  $J_0^0$  term in the text and analytically for the  $J_0^1$  term in the appendix.

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## Appendix. Equivalence of Sneddon's $J_0^1$ Term With Our New Derivation

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In this appendix, we derive an analytical equivalence between Sneddon's  $J_0^1$  term found on page 462, equation (56) of his textbook (*I*) with our own derivation, viz.,

$$J_0^1 = \frac{1 - \sqrt{R} \sin\left(\frac{\varphi}{2}\right)}{\rho} = \Im\left(\frac{\rho}{(\zeta - i)(1 + H(\rho, \zeta))}\right) = \Im\left(\frac{(\zeta - i)(H(\rho, \zeta) - 1)}{\rho}\right) \quad . \quad (\text{A-1})$$

Expanding the first term in equation A-1 using definitions  $R = \sqrt{(\rho^2 + \zeta^2 - 1)^2 + 4\zeta^2}$ , and  $(\rho^2 + \zeta^2 - 1) \tan(\phi) = 2\zeta$  yields

$$J_0^1 = \frac{1}{\rho} \left( 1 - \sqrt[4]{(\zeta^2 + \rho^2 - 1)^2 + 4\zeta^2} \sin\left(\frac{1}{2} \tan^{-1}(\rho^2 + \zeta^2 - 1, 2\zeta)\right) \right) \quad , \quad (\text{A-2})$$

and the third term in equation A-1 using  $H(\rho, \zeta) = \sqrt{1 + \frac{\rho^2}{(\zeta - i)^2}}$  yields

$$J_0^1 = \frac{1}{\rho} \Im\left(i - \zeta + \sqrt{\rho^2 + (\zeta - i)^2}\right) \quad . \quad (\text{A-3})$$

Expanding the term under the square root sign in equation A-3 and taking the imaginary part results in

$$J_0^1 = \frac{1}{\rho} \left( 1 + \sqrt[4]{(\zeta^2 + \rho^2 - 1)^2 + 4\zeta^2} \sin\left(\frac{1}{2} \arg(\rho^2 + (\zeta - i)^2)\right) \right) \quad . \quad (\text{A-4})$$

Expanding the arg function term in equation A-4 results in

$$J_0^1 = \frac{1}{\rho} \left( 1 + \sqrt[4]{(\zeta^2 + \rho^2 - 1)^2 + 4\zeta^2} \sin\left(\frac{1}{2} \tan^{-1}(\zeta^2 + \rho^2 - 1, -2\zeta)\right) \right) \quad , \quad (\text{A-5})$$

the required equation A-1 after factoring the minus sign out of the arctangent function.

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